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$\mathbb{Q}$-bases of the Néron-Severi groups of certain elliptic surfaces

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1 Introduction

P. Stiller computed the Mordell-Weil ranks and hence the Picard numbers of several families of elliptic surfaces by studying the action of certain automorphisms on the cohomology group ([Stiller 1987]). He considered five families of elliptic surfaces $\mathcal{E}_n^i$ $(1 \leq i \leq 5,\ n \in \mathbb{N})$ (Example 1, 2, 3, 4 and 5 in [Stiller 1987]). For each $\mathcal{E}_n^i$, he proved that there exists finite set Adm_i of natural numbers such that the Mordell-Weil rank $r_i(n)$ is given by

$$r_i(n) = \sum_{d|n,\ d \in \mathrm{Adm}_i} \varphi(d),$$

where φ is the Euler function. However he did not give generators of the Néron-Severi groups of these surfaces. In [Kuroda], $\mathbb{Q}$-bases or $\mathbb{Z}$-bases of these groups are given explicitly. **In this poster, we explain briefly properties with respect to such bases, and we give a basis in the most complicated case Example 1 of these five examples:**

$$\mathcal{E}_n^1 : y^2 = x^3 + t^n x + t^n \quad (n \in \mathbb{N},\ t \in \mathbb{P}^1_{\mathbb{C}}),$$
$$\mathrm{Adm}_1 = \{1,\ 2,\ 3,\ 7,\ 8,\ 10,\ 12,\ 15,\ 18,\ 20,\ 42\}.$$

2 The Néron-Severi group of an elliptic surface

Notations

$f : \mathcal{E} \to \mathbb{P}^1_{\mathbb{C}}$: an elliptic surface with a zero section
$E/\mathbb{C}(t)$: the generic fiber of $f : \mathcal{E} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$
$E(\mathbb{C}(t))$: the Mordell-Weil group of $E/\mathbb{C}(t)$
(P) : the image in $\mathcal{E}$ of the section corresponding to $P \in E(\mathbb{C}(t))$
∞ : the image of zero section
$\Sigma(\mathcal{E}) := \{t \in \mathbb{P}^1_{\mathbb{C}} \mid \mathcal{E}_t := f^{-1}(t) \text{ is a singular fiber}\}$
$F_{t,\ a}$ $(0 \leq a \leq m_t - 1)$: the irreducible components of the fiber $\mathcal{E}_t$
$F_{t,\ 0}$: the unique component of $\mathcal{E}_t$ intersecting with ∞
$C_0 := \mathcal{E}_{t_0},\ t_0 \in \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma(\mathcal{E})$: a general fiber

Properties with respect to $\mathbb{Q}$-bases of $\mathrm{NS}(\mathcal{E})$

Lemma 2.1. *Let $P_1,\ \ldots,\ P_r$ be rational points of E and let M be the intersection matrix of the associated divisors*

$$C_0,\ \infty,\ D_1,\ \ldots,\ D_r,\ F_{t,\ a} \quad (t \in \Sigma(\mathcal{E}),\ 1 \leq a \leq m_t - 1), \tag{1}$$

where $D_i = (P_i) - \infty$. Put

$$L_{t_\alpha} = (F_{t_\alpha,\ i} \cdot F_{t_\alpha,\ j})_{1 \leq i,\ j \leq m_t - 1} \text{ and } N = ((D_i + \Phi_i) \cdot D_j)_{1 \leq i,\ j \leq r}, \tag{2}$$

where $\{t_1, \ldots, t_s\} = \{t \in \Sigma(\mathcal{E}) \mid m_t \geq 2\}$ and

$$\Phi_i := \sum_{t \in \mathbb{P}^1_{\mathbb{C}}} \sum_{k=0}^{m_t-1} a_{t,\ k} F_{t,\ k} = \sum_{t \in \{t_1,\ \ldots,\ t_s\}} \sum_{k=1}^{m_t-1} a_{t,\ k} F_{t,\ k},$$
$$(a_{t,\ 1},\ \ldots,\ a_{t,\ m_t-1}) = -(D_i \cdot F_{t,\ 1},\ \ldots,\ D_i \cdot F_{t,\ m_t-1}) L_t^{-1}.$$

Then we have

$$\det(M) = -\det(N) \prod_{\alpha=1}^{s} \det(L_{t_\alpha}).$$

Remark 2.2. (i) $P_1,\ \ldots,\ P_r$ form a $\mathbb{Z}$-basis of $E(\mathbb{C}(t))_{\text{free}}$
$\Rightarrow$ (1) form a $\mathbb{Z}$-basis of $\mathrm{NS}(\mathcal{E})_{\text{free}}$ ([Shioda 1972]).

(ii) In [Shioda 1990], $(P) - \infty + \Phi_P$ is denoted by D_P. Then the pairing $\langle P,\ Q \rangle := -D_P \cdot D_Q = -((P) - \infty + \Phi_P) \cdot ((Q) - \infty)$ is *the height pairing.*

Properties

The divisors $C_0,\ \infty,\ D_j,\ F_{t,\ a}$ form a $\mathbb{Q}$ (resp. $\mathbb{Z}$)-basis of $\mathrm{NS}(\mathcal{E})$
$\iff \det(M) \neq 0$
$\iff \det(N) \neq 0$ ($\because \det(L_{t_\alpha}) \neq 0 \quad (1 \leq \alpha \leq s)$)
(resp. $\iff |\det(M)| = \det(\mathrm{NS}(\mathcal{E}))$)

3 $\mathbb{Q}$-bases of $\mathrm{NS}(\mathcal{E}_n^1)$

We give $r_1(n)$ rational points of the generic fiber $E_n^1/\mathbb{C}(t)$ of $\mathcal{E}_n^1$, and calculate the determinant of the intersection matrix M_n of the associated divisors (or $\det(N_n)$, where N_n is similar to (2)).

Definition 3.1. $\mathbb{C}(t)$-rational points $P_{d,\ j}$ of E_d^1 $(d \in \mathrm{Adm}_1)$

$j \in (\mathbb{Z}/d\mathbb{Z})^\times$
$\zeta_d = \exp\left(2\pi\sqrt{-1}/d\right)$
$[*]$: the Gauss symbol

$$P_{1,\ 1} = (-1,\ \sqrt{-1}),$$
$$P_{2,\ 1} = \left(\sqrt{-1}t,\ -t\right),$$
$$P_{3,\ j} = \left(-\zeta_3^j t,\ \sqrt{-1}\zeta_3^{2j} t^2\right),$$
$$P_{7,\ j} = \left(-\zeta_7^{2j} t^2 - \zeta_7^{3j} t^3,\ \sqrt{-1}\left(\zeta_7^{3j} t^3 + \zeta_7^{4j} t^4 + \zeta_7^{5j} t^5\right)\right),$$
$$P_{10,\ j} = \left(2^{\frac{2}{5}} \zeta_{10}^{4j} t^4,\ -\zeta_{10}^{5j} t^5 - 2^{\frac{1}{5}} \zeta_{10}^{7j} t^7\right),$$
$$P_{15,\ j} = \Big(-\zeta_{15}^{5j} t^5 - 3^{\frac{1}{5}} \zeta_{15}^{6j} t^6 - 3^{\frac{2}{5}} \zeta_{15}^{7j} t^7,$$
$$\sqrt{-1}\left(3^{\frac{3}{5}} \zeta_{15}^{8j} t^8 + 3^{\frac{4}{5}} \zeta_{15}^{9j} t^9 + 2\zeta_{15}^{10j} t^{10} + 3^{\frac{1}{5}} \zeta_{15}^{11j} t^{11}\right)\Big),$$
$$P_{d,\ j} = \left(\sum_{k=0}^{2} a_k(d,\ j) t^{2[\frac{d}{6}]+k},\ \sum_{k=0}^{3} b_k(d,\ j) t^{3[\frac{d}{6}]+k}\right) \quad (d = 8,\ 12),$$
$$P_{d,\ j} = \left(\sum_{k=0}^{2} a_k(d,\ j)(\zeta_d^j t)^{6+2k},\ \sum_{k=0}^{3} b_k(d,\ j)(\zeta_d^j t)^{9+2k}\right) \quad (d = 18,\ 20),$$
$$P_{42,\ j} = \left(\sum_{k=0}^{4} a_k(42,\ j)(\zeta_{42}^j t)^{14+2k},\ \sum_{k=0}^{6} b_k(42,\ j)(\zeta_{42}^j t)^{21+2k}\right).$$

$a_k(d,\ j),\ b_k(d,\ j)$ satisfy the system given by comparing the coefficients of

$$y(P_{d,\ j})^2 = x(P_{d,\ j})^3 + t^d x(P_{d,\ j})^2 + t^d.$$

Remark 3.2. (i) $f : \mathcal{E}_n^1 \to \mathbb{P}^1_{\mathbb{C}}$ has a global section and is not smooth.
$\rightsquigarrow \mathrm{NS}(\mathcal{E}_n^1)$ is torsion free ([Shioda 1990]).

(ii) $\rho : E_n^1 \to E_d^1$; $(x,\ y,\ t) \mapsto (x,\ y,\ t^{\frac{n}{d}})$: surjective map $(d \mid n)$
$\rightsquigarrow E_d^1(\mathbb{C}(t)) \hookrightarrow E_n^1(\mathbb{C}(t))$; $P_{d,\ j} \mapsto \rho^*(P_{d,\ j})$

We use the same symbol $P_{d,\ j}$ for $\rho^*(P_{d,\ j})$.

(iii) $\mathcal{E}_n^1$: rational (i.e., $n = 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8$ or 12)
$\Rightarrow E_n^1(\mathbb{C}(t))$ is generated by points of the form

$$x = t^{2[\frac{n}{6}]} \sum_{k=0}^{2} a_k t^k,\ y = t^{3[\frac{n}{6}]} \sum_{k=0}^{3} b_k t^k \quad \text{([Shioda 1990]).}$$

(iv) We can choose the coefficients $a_k(d,\ j),\ b_k(d,\ j)$ such that $\det(N_n) \neq 0$, and $|\det(M_n)| = 1$ if $n = 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 12$.
(In [Kuroda], these coefficients are given explicitly.)

Theorem 3.3.

(i) *$\mathrm{NS}(\mathcal{E}_n^1)$ has a $\mathbb{Q}$-basis $C_0,\ \infty,\ D_{d,\ j},\ F_{t,\ a}$*
$(d \in \mathrm{Adm}_1,\ d \mid n,\ j \in (\mathbb{Z}/d\mathbb{Z})^\times,\ t \in \Sigma(\mathcal{E}_n^1), 1 \leq a \leq m_t - 1)$.
(ii) *$\mathcal{E}_n$ is rational (i.e., $n = 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8$ or 12)*
$\Rightarrow$ *these divisors form a $\mathbb{Z}$-basis of* $\mathrm{NS}(\mathcal{E}_n^1)$.

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